

ON HIGHER GRAPH MANIFOLDS

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ABSTRACT. In this short note we introduce a general notion of higher graph manifolds and use a version of the barycenter technique to characterize when they undergo volume collapse. In the case when the pure pieces are hyperbolic, we compute the exact value of the minimal volume. We verify the Baum–Connes conjecture for these manifolds and show that they do not admit positive scalar curvature metrics. In the case without any pure pieces, we show the Yamabe invariant vanishes.

1. INTRODUCTION

Consider the class of manifolds we term *higher graph manifolds*:

Definition 1. *A compact smooth n -manifold M , $n \geq 3$, is a higher graph manifold provided that it can be constructed in the following way:*

- (1) *For every $i = 1, \dots, r$ take a complete finite-volume non-compact pinched negatively curved n_i -manifold V_i , where $2 \leq n_i \leq n$.*
- (2) *Denote by M_i the closed manifold with boundary obtained by “truncating the cusps” of V_i , i.e. by removing from V_i a (nonmaximal) horospherical open neighborhood of each cusp.*
- (3) *Take fiber bundles $Z_i \rightarrow M_i$, which are products in a neighborhood of ∂M_i and with fiber an infranilmanifold N_i of dimension $n - n_i$, i.e. N_i is diffeomorphic to \widetilde{N}_i/Γ_i , where \widetilde{N}_i is a simply connected nilpotent Lie group and Γ_i is an extension of a lattice $L_i \subset \widetilde{N}_i$ by a finite group. So the finite cover N'_i of N_i with covering group $H_i = \Gamma_i/L_i$ is a nilmanifold \widetilde{N}_i/L_i . Assume that the structure group of the bundle $Z_i \rightarrow M_i$ reduces to a subgroup of affine transformations of N_i .*
- (4) *Fix a complete pairing of diffeomorphic boundary components between distinct Z_i ’s, provided one exists, and glue the paired boundary components using diffeomorphisms smoothly isotopic¹ to affine diffeomorphisms of the boundaries, so as to obtain a connected manifold of dimension n .*

We will call the Z_i ’s the pieces of M and whenever $\dim(M_i) = n$, then we say $Z_i = M_i$ is a pure piece.

By [16] the group of connection preserving or affine transformations, $\text{Aff}(N)$, of a compact infranilmanifold $N = G/\Gamma$ is precisely $C(G)/(C(G \cap \Gamma) \rtimes \text{Aut}(\Gamma))$. (Here $C(X)$ stands for the center of X .) It can easily be computed that $\pi_k(B \text{Aff}(N))$ is always finitely generated, but $\pi_k(\text{Diff}(G/\Gamma))$ can be infinitely generated in some cases. Hence, not all bundles above arise with the given structure group condition, apart from the collared restriction.

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¹We rely on some results not available for smooth pseudo-isotopies

Remark 1. We wish to point out that the higher graph manifolds includes the class of generalized graph manifolds introduced in the recent paper by Frigerio, Lafont and Sisto [10]. In their definition, the pieces V_i in item (1) above are required to be hyperbolic with toral boundary cusps, the N_i in item (3) are required to be tori, and the gluing diffeomorphisms in item (4) are required to be affine diffeomorphisms.²

Remark 2. The higher graph manifolds do not strictly generalize the notion of graph manifold in dimension 3 since the latter are permitted to have stand-alone Seifert-fibered pieces such as a 3-torus. Conversely, a graph manifold in dimension 3 does not contain hyperbolic pieces that are permitted in these higher graph 3-manifolds.³

Remark 3. We also point out that in [10] the authors produce some simple and not-so-simple examples of higher graph manifolds that do not admit any $CAT(0)$ metric. For example, using only pieces that are products of hyperbolic manifolds and tori they exhibit examples of higher graph manifolds whose fundamental groups have non-quasiconvex \mathbf{Z}^2 subgroups which $CAT(0)$ groups cannot have. They also produce more sophisticated examples that fail to be $CAT(0)$ for more subtle reasons. Indeed, such examples highlight the increased complexity arising from boundary gluing diffeomorphisms which are isotopic to affine maps instead of just isometries.

Now we present our main results. The minimal volumes Minvol and Vol_K are closely related smooth topological invariants of M defined as [12],

$$\text{Minvol}(M) = \inf \{ \text{Vol}(M, g) \mid -1 \leq K_g \leq 1 \}$$

and

$$\text{Vol}_K(M) = \inf \{ \text{Vol}(M, g) \mid -1 \leq K_g \},$$

where K_g represents the sectional curvatures of g .

Theorem 1. Suppose that M is a higher graph manifold such that there are no pure pieces, then M collapses. In particular, $\text{Minvol}(M) = \text{Vol}_K(M) = 0$.

An important property of Minvol is that it is capable of distinguishing between different smooth structures on an underlying topological manifold. This was first shown by Bessi eres in [5] and there are known examples of manifolds where even vanishing of Minvol depends on the smooth structure.

For a subclass of higher graph manifolds constructed as doubles of hyperbolic manifolds P. Ontaneda showed the existence of various distinct interesting smooth structures (some admit nonpositively curved metrics and others do not) [20].

The simplicial volume $||M||$ of a closed orientable manifold M is defined as the infimum of $\sum_i |r_i|$ where r_i are the coefficients of any real singular cycle representing the fundamental

² **Remark 1** $\frac{1}{2}$. The family of manifolds called cusp-decomposable manifolds studied by T. Tam Nguyen Phan in [26]—where interesting (non)rigidity properties are explored—is also included in these higher graph manifolds. Seen as higher graph manifolds they have only pure pieces.

Remark 1 $\frac{3}{4}$. Higher graph manifolds also include affine twisted doubles of hyperbolic manifolds. C.S. Aravinda and T. Farrell study in [1] existence of nonpositively curved metrics for this class of spaces.

³**Remark 2** $\frac{1}{2}$. In fact F. Waldhausen [29] classified graph manifolds in dimension 3. The construction in [29] justified the use of the term **graph**, just like in our case here. However, we only consider nonsingular bundles.

class of M . This invariant was introduced by M. Gromov in order to find lower bounds for the minimal volume.

For oriented connected compact n -manifolds with boundary, the relative simplicial volume $\|Z_i, \partial Z_i\|$ is defined as the ℓ^1 -semi-norm on relative singular homology $H_n(Z_i, \partial Z_i; \mathbf{R})$ of the relative fundamental class $[Z_i, \partial Z_i] \in H_n(Z_i, \partial Z_i; \mathbf{R})$.

Our second theorem covers the converse to Theorem 1.

Theorem 2. *For any higher graph manifold M , the simplicial volume satisfies*

$$\|M\| = \sum_i \|Z_i, \partial Z_i\|.$$

In particular it is positive if there is one pure piece.

The *volume growth entropy* $h(g)$ of any manifold (M, g) is the quantity

$$h(g) = \limsup_{R \rightarrow \infty} \frac{\log \text{Vol } B(x, R)}{R},$$

where $B(x, R)$ is the geodesic ball of radius R in the universal covering \tilde{M} . Whenever M covers a compact manifold, the limsup in the definition of $h(g)$ may be replaced by a limit [19].

Similarly, we define the *minimal volume entropy* of any manifold M as

$$h(M) = \inf \{h(g) \mid \text{Vol}(M, g) = 1\}.$$

Our next theorem gives bounds for Minvol and Vol_K and it relates them to the minimal volume entropy.

Theorem 3. *Let M be a higher graph manifold all of whose pure pieces M_1, \dots, M_k are locally symmetric, and let $M_{\text{sym}} = \coprod_{i=1}^k V_i$, where V_i is the interior of M_i . The V_i are all quotients of the same symmetric space, and if g_{sym} represents the symmetric metric on M_{sym} normalized to have sectional curvatures between -4 and -1 , then*

$$h_0 \text{Vol}(M_{\text{sym}}, g_{\text{sym}})^{\frac{1}{n}} \leq h(M) \leq 2(n-1) \text{Vol}(M_{\text{sym}}, g_{\text{sym}})^{\frac{1}{n}}$$

where $h_0 \in \{n-1, n, n+2, 22\}$ depending on whether all of the pure pieces are of real hyperbolic, complex hyperbolic, quaternionic hyperbolic, or Cayley hyperbolic type. Moreover,

$$\frac{h_0^n}{(n-1)^n} \text{Vol}(M_{\text{sym}}, g_{\text{sym}}) \leq \text{Vol}_K(M) \leq \text{Minvol}(M) \leq 2^n \text{Vol}(M_{\text{sym}}, g_{\text{sym}}).$$

(If $k = 0$ we interpret all quantities as 0.)

We can give the exact value for $\text{Minvol}(M)$ when the pure pieces are hyperbolic.

Theorem 4. *With the notation of the previous theorem, when the pure pieces of M are hyperbolic, the minimal volume of M is precisely*

$$\text{Minvol}(M) = \text{Vol}_K(M) = \text{Vol}(M_{\text{sym}}, g_{\text{sym}}) = \sum_{i=1}^k \text{Vol}(V_i, g_{\text{hyp}})$$

where g_{hyp} represents the complete finite volume constant curvature -1 metric on each V_i .

This answers a problem posed in [10] positively.

Remark 4. *Even when the pure pieces are hyperbolic we do not know if $h(M)$ is exactly achieved by the locally symmetric metric, because the entropy is not necessarily continuous with respect to the Gromov-Hausdorff topology.*

Consider the reduced C^* -algebra $C_r^*(G)$ of a group G . The Baum-Connes conjecture proclaims that the complex K -homology of the classifying space BG is isomorphic to the topological K -theory of $C_r^*(G)$. Using existing results, we are able to verify the Baum-Connes conjecture for these manifolds.

Proposition 5. *Let M be a compact higher graph manifold, then $\pi_1(M)$ satisfies the Baum-Connes conjecture.*

We take this opportunity to record two corollaries that follow from the validity of the Baum-Connes conjecture for these groups (see [27]). For a connected, closed, oriented, smooth manifold M its Hirzebruch L -class L_M is a polynomial in the Pontrjagin classes of M . The Novikov conjecture states that, for the map $f : M \rightarrow B(\pi_1 M)$ that classifies the universal cover, the class $f_*(L_M) \in H_*(B(\pi_1 M), \mathbf{Q})$ is an oriented homotopy invariant. Equivalently, the higher signatures $\sigma_u(M) \in \mathbf{Q}$, defined by $\sigma_u(M) = \langle L_M \cup f^*(u), [M] \rangle$ are, for all $u \in H^*(B(\pi_1 M), \mathbf{Q})$, oriented homotopy invariants.

Corollary 6. *Let M be an oriented compact higher graph manifold, then the Novikov conjecture holds for M .*

In fact, Proposition 5 actually implies the (so-called) *strong* Novikov conjecture, which in turn has been shown by J. Lott in [18] to imply that such a manifold has a *zero in the spectrum*:

Corollary 7. *Let M be an oriented compact higher graph manifold, then there exists a $p \geq 0$, such that zero belongs to the spectrum of the Laplace-Beltrami operator Δ_p acting on square-integrable p -forms on the universal cover of M .*

Applying results about asymptotic dimension of groups and the fact that higher graph manifolds are aspherical, we are able to show:

Theorem 8. *Let M be a compact higher graph manifold. Then M does not admit any smooth metric of positive scalar curvature.*

Consider a fixed conformal class γ of smooth metrics g on the smooth closed manifold M , and let the Yamabe constant of (M, γ) be

$$\mathcal{Y}(M, \gamma) = \inf_{g \in \gamma} \frac{\int_M s_g d\text{vol}_g}{(\text{Vol}(M, g))^{1-2/n}}.$$

Here s_g represents the scalar curvature of g . The *Yamabe invariant* is then defined to be $\mathcal{Y}(M) = \sup \mathcal{Y}(M, \gamma)$, where the supremum is taken over all conformal classes of metrics on M . As a consequence we derive that the Yamabe invariant vanishes when all the pieces have infranilpotent fibers of positive dimension:

Corollary 9. *Let M be a compact higher graph manifold without pure pieces. Then the Yamabe invariant of M equals 0.*

The proofs of all of the results are found in the following section.

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2. PROOFS

In our following proof of Theorem 1 we will adapt arguments first explained by K. Fukaya in [11], where he constructed an explicit sequence of volume collapsing metrics on fibre bundles with infranilpotent fibres and affine structure group.

Briefly, Fukaya's insight was to use the Lie algebra structure to organize charts of Z_i and describe a collapsing sequence of metrics using the center of the Lie algebra $\tilde{\mathfrak{n}}$ of the nilpotent Lie group \tilde{N} that covers the fiber N . First we choose a connection relative to the structure group of the fibration. A metric is then defined using the horizontal and vertical subspaces set by the connection and, crucially, the central direction of $\tilde{\mathfrak{n}}$, which is the part of the metric that collapses. As the structure group is affine, it is possible to construct charts compatible with this metric, which moreover keep the collapsing directions of the metric consistent (since they come from central elements, and the charts are glued using affine transformations). So we obtain a parametrised family of smooth metrics on all of Z_i whose volume vanishes in the limit (in fact the base M_i is the Gromov-Hausdorff limit). The curvature is shown to be bounded throughout this process.

Proof of Theorem 1. The explicit construction of a volume collapsed sequence of metrics for fibre bundles with infranilpotent fibre of Theorem 0-7 in [11] applies directly to each of the pieces Z_i that make up M . It is enough to show that these metrics can be used to provide a sequence of volume collapsing metrics on the whole of M .

Notice that on each of the pieces, Z_i , Fukaya's construction details how to construct a volume collapsing sequence of metrics on the infranilmanifold N_i . By combining these into the fibered metric we obtain a sequence of volume collapsing metrics $(g_\epsilon)_i$ on Z_i . As ϵ tends to 0, the piece $(Z_i, (g_\epsilon)_i)$ collapses down to M_i .

We will see that these metrics can be patched along two contiguous pieces, so as to obtain a new sequence of volume collapsing smooth metrics. Assume Z_1 and Z_2 are two such contiguous pieces, each glued along the product of a connected component of the boundary ∂M_i of their respective M_i with the respective infranilmanifold N_i for $i = 1, 2$.

Later, we need to build explicit collapsing metrics on each of the pieces. So first we will describe how to widen the region where each boundary component is joined in order to properly interpolate these collapsing metrics.

First consider horospherical parametrizations of the truncated portion of the cusps of V_i extending from ∂M_i that glue Z_1 and Z_2 so that the negatively curved metrics in these parameters are described by a generalized warped Riemannian product on a neighborhood of each boundary component:

$$dy_{c_i(t)}^2 + dt^2$$

Here $dy_{c_i(t)}^2$ represents the metric on the cross-section at parameter t , $c_i(t) \cong C_i \times \{t\}$, of the cusp of V_i topologically identified with $C_i \times [0, \infty)$. If we represent a fixed locally

homogeneous metric on N_i by $dx_{N_i}^2$ then the metric on a neighborhood of any boundary component of Z_i can be expressed by:

$$dx_{N_i}^2 + dy_{c_i(t)}^2 + dt^2$$

We will use x for a variable point in N_i and y for a variable point in $c_i(t)$. Let p_i be the projection of Z_i onto the cusp parameter t .

Let $1 > \delta > 0$. To fix notation, assume that with respect to the t parameter, the portion of the cusp of V_1 that was truncated to obtain M_1 corresponds to $t \in (-\infty, \delta)$, and for M_2 it corresponds to $t \in (-\delta, \infty)$. If f is the gluing diffeomorphism of the ends, then let a be the affine map that f is isotopic to. Since the diffeomorphism type of the resulting glued manifold only depends the isotopy type of the gluing map, we may and will use a instead of f in all further operations.

Define A to be the union of the preimages $p_1^{-1}\{(-3, \delta)\}$ and $p_2^{-1}\{(-\delta, 3)\}$, glued along the portion corresponding to $t \in (-\delta, \delta)$. Specifically, we glue $\partial Z_1 \times (-\delta, \delta)$ to $\partial Z_2 \times (-\delta, \delta)$, by $a \times Id$. The space A has the structure of a total space of a smooth fibre bundle over $(-3, 3)$ with infranilpotent fiber ∂Z_i given by the projections p_i . Though globally trivial, this bundle is not trivializable relative to the boundary; it has relative structure group with a single monodromy map a .

Since the bundle map $\pi : A \rightarrow (-3, 3)$ satisfies the three conditions in the hypotheses of Fukaya's Theorem 0-7 from [11], we may apply this to obtain a sequence of metrics $(g_\epsilon)_A$ on A that are volume collapsing and have bounded sectional curvatures. The limit metric space of this sequence of metrics is the interval $(-3, 3)$.

The important point here is that since the structure group of the A bundle relative the boundary is affine, the proof of Fukaya's theorem shows that after the identifications in A and Z_i are made, the portion of A that is being collapsed through the metrics $(g_\epsilon)_A$, namely the quotient of the centralizer in the nilmanifold fiber $C_i \times N_i$, will at least contain the quotient of the centralizer belonging to the fiber N_i in each piece Z_i which is being collapsed through the metrics $(g_\epsilon)_i$. (This would not necessarily be the case if we were attaching the ends via an arbitrary diffeomorphism.)

At this point we collect the three collapsing metrics we have described. Put the metric

$$\beta_i(t)(g_\epsilon)_i + \beta_A(t)(g_\epsilon)_A$$

on the overlap of A and Z_i with respect to the horospherical parameter t on each C_i , where the functions β_i are smooth cut-off functions with values:

$$\begin{aligned} \beta_1(t) &= 1 & \text{for } t &\in (-\infty, -2) \\ \beta_1(t) &= 0 & \text{for } t &\in (-1, \delta) \\ \beta_2(t) &= 0 & \text{for } t &\in (-\delta, 1) \\ \beta_2(t) &= 1 & \text{for } t &\in (2, \infty) \end{aligned}$$

Similarly, the function β_A is a smooth bump function with values:

$$\begin{aligned}\beta_A(t) &= 0 & \text{for } t &\in (-\infty, -2) \\ \beta_A(t) &= 1 & \text{for } t &\in (-1, 1) \\ \beta_A(t) &= 0 & \text{for } t &\in (2, \infty)\end{aligned}$$

Suppressing the fixed $\epsilon > 0$, let g_t denote the combined metric on the fiber over t , which we recall is topologically $N_i \times C_i$, and let g be the overall metric. Now, denote the field of the t coordinate as ∂_t , and choose any g_t -orthogonal geodesic frame $\{X_i\}$ in a neighborhood of a point p in each fiber over t . Writing $h_i(r) = \sqrt{g_t(X_i, X_i)}$, the fields $Y_i = X_i/h_i$ are g_t -orthonormal. Then we can express the curvature tensor in these coordinates by (see Appendix C of [3] correcting [4]):

$$\begin{aligned}R_g(Y_i, Y_j, Y_j, Y_i) &= R_{g_t}(Y_i, Y_j, Y_j, Y_i) - \frac{h'_i h'_j}{h_i h_j} \\ R_g(Y_i, Y_j, Y_k, Y_l) &= R_{g_t}(Y_i, Y_j, Y_k, Y_l) \quad \text{if } \{i, j\} \neq \{k, l\} \\ R_g(Y_i, \partial_t, \partial_t, Y_i) &= \frac{h''_i}{h_i} \quad \text{and} \quad R_g(Y_i, \partial_t, \partial_t, Y_j) = 0 \quad \text{if } i \neq j \\ 2R_g(\partial_t, Y_i, Y_j, Y_k) &= \langle [Y_i, Y_j], Y_k \rangle \left(\log \frac{h_k}{h_j} \right)' + \langle [Y_k, Y_i], Y_j \rangle \left(\log \frac{h_j}{h_k} \right)' \\ &\quad + \langle [Y_k, Y_j], Y_i \rangle \left(\log \frac{h_i^2}{h_j h_k} \right)'\end{aligned}$$

If we let $\{X_i\}$ be a fiberwise orthogonal base for g then each X_i can be expressed linearly in terms of an orthogonal basis for $g_{i,t}$ or $g_{A,t}$, and the expression for each h_i is explicitly algebraic in terms of the h_j for each of g_i and g_A and β_i and β_A . Note that the relevant quantities involving the h_j in the expressions for the curvatures of both $g_{i,\epsilon}$ and $g_{A,\epsilon}$ in the above expressions remain bounded as ϵ tends to 0 by assumption, and that the β_i and β_A and their derivatives are bounded away from 0 and infinity in the region of the overlaps. We note that as $\epsilon \rightarrow 0$ the length of ∂_t in the transverse t direction stays universally bounded from above and below. In particular the curvatures of g_ϵ remain bounded as $\epsilon \rightarrow 0$. Lastly, since both metrics on each side shrink the common centralizer, we have produced a sequence of volume collapsing metrics with bounded curvature on the glued pieces Z_1 and Z_2 . Repeat this procedure on every pair of contiguous glued pieces of M .

In the end, the resulting metric may only be C^2 since the cuspidal Busemann functions in finite volume manifolds, which we used for the parameter t , are only C^2 in general. However, by standard approximation theory (see [15]), any C^2 atlas is compatible with a C^∞ one. In this atlas we can approximate the given metric g_ϵ with a C^∞ metric $g_{\epsilon,\eta}$ with C^2 norm $\|g_\epsilon - g_{\epsilon,\eta}\|_{C^2}$ sufficiently small so that the difference of both the sectional curvatures and the volumes are less than η . Then choosing the bound $\eta = \epsilon$, then the sequence of metrics $g_{\epsilon,\epsilon}$ is C^∞ and also collapses. \square

Notice that the limit metric space of the volume collapsing sequence of metrics we have just obtained is essentially the pinched negatively curved factors M_i of the pieces Z_i , whose cusps have been squashed into lines and joined together by intervals connecting them. This

limit metric space is also described by the graph that represents the fundamental group of M . Fukaya's original argument in [11] applies almost directly to this situation, but the dimension of the base in each of the (trivially) fibered pieces changes.

Remark 5. *It may be possible to construct an explicit polarized F -structure (or even an N -structure) on higher graph manifolds without pure pieces, but we did not succeed in this direction. This is certainly true for higher graph manifolds whose pieces are trivial bundles where the fibers are nilmanifolds. In this special case one can flesh out the compatibility conditions needed for an F -structure to exist using the affine map and the central circle actions on the fibers. Already in the case of trivial bundles with infranilmanifold fibers it is not clear how to implement this idea. Even though it seems there should be a global nilpotent structure that describes all possible collapsing directions, in the spirit of the work of Cai and Rong in [8].*

Proof of Theorem 2. For any boundary component E of any piece Z_i , we claim that the homomorphism $\pi_1(E) \rightarrow \pi_1(Z_i)$ inducted by inclusion is injective. To see this, we first note that since V_i has finite volume, the inclusion of the corresponding boundary component, $B \subset \partial M_i$, is π_1 injective in $\pi_1(M_i)$. Second, since the pair (Z_i, E) is the total space of an N_i fibration over (M_i, B) , the injectivity can be deduced from the long exact sequence.

Moreover, the inclusion is amenable simply because $\pi_1(E)$ is amenable. Gromov's *cutting-off* theorem of [12] has been extended by T. Kuessner in his thesis [17] to exactly cover the case in question. By Lemma 11 (i),(ii) of [17], we obtain that $\|M, \partial M\| \geq \|Z_i, \partial Z_i\| + \|Q, \partial Q\|$ where Z_i is the one piece and Q is the complementary manifold. (Lemma 11 (ii) covers the case where two boundary components of M_i are glued together.) Inductively, we obtain $\|M\| \geq \sum_i \|Z_i, \partial Z_i\|$.

For the reverse inequality, we may apply Lemma 12 in the same thesis since the fundamental group of the entire collar is amenable.

In [12] Gromov proved the positivity of $\|M_i, \partial M_i\|$ whenever $\text{int}(M_i) = V_i$ admits a finite volume metric of pinched negative curvature. The non pure pieces collapse and therefore the simplicial volume is at least the sum of the relative simplicial volumes of the pure pieces which is positive. \square

The following proposition generalizes part of Theorem 4 of J. Souto [25]. We closely follow the proof found there. We will need this preliminary result to prove Theorem 4.

Proposition 10. *Let M be a closed higher graph n -manifold whose pure pieces are all locally symmetric of the same type. If $(M_{\text{sym}}, g_{\text{sym}})$ represents a locally symmetric metric on the pure pieces scaled to have curvatures between -1 and -4 , then*

$$h(M) \geq h_0 (\text{Vol}(M_{\text{sym}}, g_{\text{sym}}))^{\frac{1}{n}}$$

where $h_0 \in \{n-1, n, n+2, 22\}$ depending on whether all of the pure pieces are of real hyperbolic, complex hyperbolic, quaternionic hyperbolic, or Cayley hyperbolic type.

Proof. We first consider a CW-complex X defined as the quotient space formed by taking M and collapsing each embedded non-pure piece $Z_i \subset M$ to a point and also collapsing a small product neighborhood of each component B_j of $\partial M_i \subset M$ to a distinct point p_j where $p_j = p_l$ only if the component $B_j \subset M_i$ is identified with $B_l \subset M_r$. The space X has a naturally associated graph structure \mathcal{G} with vertices being the special points p_1, \dots, p_m , and an edge

between p_i and p_j whenever the projection of some $M_l \subset M$ has boundary containing p_i and p_j in X .

Let $f : M \rightarrow X$ be the quotient map. Note that

$$f_*[M] = [M_1, \partial M_1] + \cdots + [M_k, \partial M_k] \in H_n(X, \mathbf{Z}).$$

Let $\mathcal{S} = \{p_1, \dots, p_m\}$. We observe that $\mathcal{R} = X - \mathcal{S}$ is homeomorphic to $\coprod_{i=1}^k V_i$. Starting with locally symmetric metrics on each V_i , we can smoothly deform the metric in a neighborhood of each end with an incomplete warped product metric which is completed by a single cusp point. This (highly) negatively curved metric induces a locally CAT(-1) metric on all of X , which is smooth off of \mathcal{S} .

We now describe the family of natural maps $F_s : M \rightarrow X$ homotopic to f and estimate the Jacobian on $F_s^{-1}(\mathcal{R})$.

Let $U = f^{-1}(\mathcal{S})$ and $\tilde{U} \subset \tilde{M}$ be the preimage of U in the universal cover of M . For all $s > h(g)$ consider the measure μ_x^s supported on \tilde{U} absolutely continuous with Lebesgue measure and with Radon-Nikodym derivative

$$\frac{d\mu_x^s}{d\text{Vol}_{\tilde{M}}}(z) = \chi_{\tilde{U}} e^{-sd(x,z)}.$$

Where d is the distance on \tilde{M} . Note that the measure has finite total mass by the condition on s .

Set $\sigma_x^s = f_*\mu_x^s$. Now consider the function $\mathcal{B}_{s,x} : \tilde{X} \rightarrow \mathbf{R}$ given by

$$\mathcal{B}_{s,x}(y) = \int_{\tilde{X}} d_{\tilde{X}}(y, z) - d_{\tilde{X}}(x, z) d\sigma_x^s(z).$$

Note that the distance function on the CAT(-1) space \tilde{X} is smooth when restricted to each component of \tilde{R} , and any geodesic joining distinct components of \tilde{R} passes through points of \tilde{S} . It follows that $d_{\tilde{X}}$ is smooth at each pair of distinct points $z \in \tilde{X}$ and $y \notin \tilde{S}$. In particular, since the support of σ_x^s is \tilde{S} , $\mathcal{B}_{s,x}$ is smooth on \tilde{R} . Moreover, it is strictly convex and we denote its unique minimum point by $\text{Bar}(\sigma_x^s)$. We now set $\tilde{F}_s(x) = \text{Bar}(\sigma_x^s)$. Since this map is equivariant under the actions of $\pi_1(M)$ and $\pi_1(X)$, it descends to a continuous map $F_s : M \rightarrow X$.

It is easy to verify that setting $\tilde{\Psi}_t(x) = \text{Bar}(t\delta_{\tilde{f}(x)} + (1-t)\sigma_x^s)$ produces an explicit equivariant homotopy from $\tilde{F}_s = \tilde{\Psi}_0$ to $\tilde{f} = \tilde{\Psi}_1$.

We now rely on a large scale local version of a key global estimate originally found in [7].

Lemma 11. *For all $s > h(g)$, the natural map $F_s : M \rightarrow X$ is C^1 on $F_s^{-1}(\mathcal{R})$. Moreover, for any $\epsilon > 0$ there is an $R_\epsilon > 0$, with $\lim_{\epsilon \rightarrow 0} R_\epsilon = \infty$, such that for any component $V_i \subset \mathcal{R}$ and any $x \in F_s^{-1}(V_i)$ whose ball $B(F_s(x), R_\epsilon)$ in V_i is isometric to that of a rank one locally symmetric space of with entropy h_i , then*

$$|\text{Jac } F_s(x)| \leq (1 + \epsilon) \left(\frac{s}{h_i} \right)^n.$$

A proof of the above lemma in the case when the comparison locally symmetric space is hyperbolic, can be found in the proof of Proposition 4' in [25], except that 3 must be replaced by n . For the remaining three types of symmetric spaces, the proof is essentially identical, except that the estimate input from Lemma 11 of [25] must be changed to account for the

complex structures in precisely the same way it is in Appendix B of [7] (see Lemma 5.5 of [6]). This relies also on the fact that balls in the other symmetric rank one spaces also tend to horospheres as their centers and radii tend to infinity at the same rate.

Now we construct a sequence of incomplete metrics g_i^j on V_i which are locally symmetric on a set of diameter at least j , and whose remaining set has volume ϵ . We achieve this by taking a sequence of successively larger diameter locally symmetric metrics on M_j whose injectivity radius of each boundary component tends to 0. For each such metric we then cap each end off as before with a negatively curved warped product metric cusp of finite diameter. Since the boundary components of the locally symmetric portion may be chosen with arbitrarily small injectivity radius, and hence $n - 1$ volume, the total volume of the non-locally symmetric finite cusps may be chosen to have total volume less than ϵ .

Therefore, for each $\epsilon > 0$ and constant R_ϵ from Lemma 11 we choose $j > R_\epsilon$ and thus we can apply Lemma 11 to the component V_i with metric g_i^j . Putting these together we have metrics g_j on all of X , and a locally symmetric submanifold $(X_j, g_j) \subset (X, g_j)$ with $\text{Vol}(X - X_j, g_j) < k\epsilon$. Hence,

$$\text{Vol}(X_j, g_j) \leq \int_{F_s^{-1}(X_j)} |\text{Jac } F_s| d\text{Vol}_g \leq (1 + \epsilon) \sum_{i=1}^k \left(\frac{s}{h_i}\right)^n \text{Vol}(M_i, g),$$

and adding the remainder yields:

$$\text{Vol}(X, g_j) \leq 2k\epsilon + \text{Vol}(X_j, g_j) \leq 2k\epsilon + (1 + \epsilon) \sum_{i=1}^k \left(\frac{s}{h_i}\right)^n \text{Vol}(M_i, g).$$

Taking $\epsilon \rightarrow 0$, and thus $j \rightarrow \infty$, we obtain that $\text{Vol}(X_j, g_j)$ tends to $\text{Vol}(M_{\text{sym}}, g_{\text{sym}})$. Moreover, since all of the pure pieces of a graph manifold must be of the same rank one locally symmetric type so the h_i 's are all equal to a single value h_0 . Therefore, we obtain

$$\text{Vol}(M_{\text{sym}}, g_{\text{sym}}) \leq \frac{s^n}{h_0^n} \sum_{i=1}^k \text{Vol}(M_i, g) \leq \frac{s^n}{h_0^n} \text{Vol}(M, g).$$

taking $s \rightarrow h(g)$ from above yields the result.

$$h_0^n \text{Vol}(M_{\text{sym}}, g_{\text{sym}}) \leq h(g)^n \text{Vol}(M, g).$$

□

Proof of Theorem 3. We first explain why, under the assumptions, all of the pure pieces must be of the same type, that is, covered by the same locally symmetric space. For each pure piece $Z_i = M_i$ that is not hyperbolic, each component of ∂M_i is a compact quotient of the same nilpotent group, either \mathbf{R}^n in the real hyperbolic case, or else a group of generalized Heisenberg type. These quotients, with the exception of tori in the real hyperbolic case, are irreducible and in particular not diffeomorphic to any nontrivial product manifold. Therefore, since we assume all of the pure pieces are locally symmetric, as soon as one of the pure pieces is not real hyperbolic, then it can only be glued to a pure piece and ones that are covered by the same symmetric space, and hence have the same type of nilmanifold boundary components. (If all the pure pieces are real hyperbolic, then these may be glued to non pure pieces.) Note that it is essential that we assume that all pure pieces are locally symmetric.

This is because a single pure piece that is not locally symmetric may have distinct boundary components which are quotients of different nilpotent groups.

The conclusion of Proposition 10 then gives the lower bound of the first inequality of the entropy estimate. If we assume that the curvatures of (M, g) are scaled to have lower bound -4 , then $h(g) \leq 2(n-1)$ by the Bishop-Gromov comparison theorem. In summary,

$$h_0 \operatorname{Vol}(M_{sym}, g_{sym})^{\frac{1}{n}} \leq h(g) \operatorname{Vol}(M, g)^{\frac{1}{n}} \leq 2(n-1) \operatorname{Vol}(M, g)^{\frac{1}{n}}.$$

Now observe that the quantity $h(g) \operatorname{Vol}(M, g)^{\frac{1}{n}}$ is homothety invariant. Hence, restricting to metrics within the given smooth class, we may express $h(M)$ as,

$$h(M) = \inf_g \left\{ h(g) \operatorname{Vol}(M, g)^{\frac{1}{n}} \right\} = \inf_g \left\{ h(g) \operatorname{Vol}(M, g)^{\frac{1}{n}} \mid -4 \leq K_g \right\}.$$

We now produce a sequence of metrics g_j on M with $K_{g_j} \geq -4$ whose volumes converge to $\operatorname{Vol}(M_{sym}, g_{sym})$. If entropy were Gromov-Hausdorff continuous then this would show that $h(M) = h_o \operatorname{Vol}(M_{sym}, g_{sym})^{\frac{1}{n}}$. However, since this is unknown, we will be content using the bound $h(g_j) \leq 2(n-1)$.

To begin with, we note that boundary horospherical nilmanifolds of the locally symmetric pieces are either flat in the real hyperbolic case, or else have a generalized Heisenberg type with intrinsic curvatures between -4 and 4 when the locally symmetric metric has curvature bounds in $[-4, -1]$. We now show that there is a sequence of metrics g_j of curvature bounded between -4 and 4 on M such that $\operatorname{Vol}(M, g_j) \rightarrow \operatorname{Vol}(M_{sym}, g_{sym})$.

Given a $j \in \{1, 2, \dots\}$, we construct the metrics g_j on M as follows. Start with a locally symmetric metric on each piece M_i with curvatures in $[4, -1]$, horospherical boundary components and extending sufficiently far out each cusp such that each of the boundary components of M_i have volume e^{-j-10} . On a width 1 collar of each component of ∂M_i we alter the locally symmetric metric in the following way. Starting from the boundary, a convex horospherical nilmanifold Q , and using a warped product metric $dt^2 + f(z, t)dz^2$ on $[0, 1] \times Q$ we can choose $f(z, t)$ to transition the metrics on the slices for $t \in [0, 1]$ from the nilpotent metric on $\{0\} \times Q$ inherited from the locally symmetric one to one on $\{1\} \times Q$ that is totally geodesic. To do this $f(z, t)$ starts as the proper nilpotent dilatation of $f(z, 0)$ for t near $t = 0$ and decreases with the second derivative $\partial_{tt}f(z, t) > 0$ until made to be constant in t in a neighborhood of $t = 1$. (Note that here t does not correspond to the same horospherical parameter used in earlier constructions.)

For real hyperbolic pieces, the boundary components in the induced metric above are totally geodesic flat manifolds, and hence the total curvature of these metrics lie in the interval $[-1, 0]$. For a non-hyperbolic symmetric space with metric normalized to lie in the interval $[-4, -1]$, the induced metric on the generalized Heisenberg group lies between $[-\frac{1}{4}, \frac{3}{4}]$. Therefore in this case, the restriction of the metrics g_j to such a piece has curvatures lying in $[-4, \frac{3}{4}]$.

The non-pure pieces combine together to form connected components which will glue to the symmetric pieces. Each of these components admits a fibered metric with structure of a nilmanifold bundle over a negatively curved manifold. Again since these have volume collapsing metrics of bounded curvature in $[-1, 1]$ such that the total volume is at most e^{-j} . Without loss of generality we may assume that the boundary manifolds of these components remain metrically nilmanifolds covered by Euclidean space or else generalized Heisenberg type groups, but they may have undergone a dilation or other affine transformation.

The gluing diffeomorphisms of the pure pieces will be isotopic to affine maps which are induced from an automorphism of the underlying nilpotent Lie group. Hence, even if the boundaries in these metrics were totally geodesic, we still cannot not directly join these metrics on the pieces together unless the affine map is an isometry. Instead, we will proceed similarly to the proof of Theorem 1.

If $N = \partial Z_1 = \partial Z_2$ is the boundary nilmanifold with nilpotent universal cover \tilde{N} between two paired boundary components Z_1 and Z_2 , then we will build a patch metric on $A = [-3, 3] \times N$ so that the manifolds may be glued isometrically to the boundaries of the patch in such a way that the resulting manifold will be diffeomorphic to the original higher graph manifold.

Reparameterizing t in the construction of the metrics on a neighborhood of Z_i given above, we assume the metric $f(z, t) = f(z)$ is constant in t for $t \in [-3, -2]$ on Z_1 and constant in t for $t \in [2, 3]$ on Z_2 . Proceeding as before, we build collapsing metrics on A whose collapsing central directions in each fiber coincide with the collapsing directions in the portion of the fibered metric $f(z, t) = f(z)$ where they are averaged together using bump functions for $t \in [-3, -2]$ and $t \in [2, 3]$. We assume the fiber metrics here are sufficiently small so that the volume of all of A is less than e^{-j} .

By construction, away from the portion of (M, g_j) which carries a locally symmetric metric, the volume of g_j is bounded by a constant times e^{-j} as was just explained above. (Recall that all the non-symmetric collars and attaching tubes have uniformly bounded transversal length, and the connected components of attached nonpure pieces have volumes at most e^{-j} .) Taking a limit as $j \rightarrow \infty$, the sequence of metrics (M, g_j) has volumes converging to $\text{Vol}(M_{\text{sym}}, g_{\text{sym}})$. Since the curvature of the metrics is bounded below by -4 , the entropy estimate mentioned at the beginning gives us the upper bound of the first inequality. \square

Remark 6. *It should be noted that in the above proof, even in the case when the pure pieces are all real hyperbolic, we cannot necessarily construct global metrics of curvature lying in $[-1, 0]$. In [10] the authors construct some simple and not-so simple examples of higher graph manifolds, though without pure pieces, which admit no $\text{CAT}(0)$ metric.*

Proof of Theorem 4. Note that under the hypothesis of hyperbolic pure pieces, the metrics g_j on M constructed in the proof of Theorem 3 have curvatures lying in the interval $[-1, 0]$, since the boundary infranilmanifolds in this case are just flat manifolds, though not necessarily tori. (The non-pure pieces cannot have any non-flat infranil-factors if there is one pure hyperbolic piece.)

Since $\text{Vol}(M, g_j)$ tends to $\text{Vol}(M_{\text{hyp}}, g_{\text{hyp}})$, combining this with the second inequality of Theorem 3 completes the proof.

For the second statement of the theorem, if (M, g) has curvature bounds $K_g \geq -1$, then $h_0 \geq n - 1 \geq h(g)$ and from the lower bound of the first statement we have,

$$\frac{h_0}{n-1} \text{Vol}(M_{\text{sym}}, g_{\text{sym}})^{\frac{1}{n}} \leq \frac{h_0}{h(g)} \text{Vol}(M_{\text{sym}}, g_{\text{sym}})^{\frac{1}{n}} \leq \frac{h(M)}{h(g)} \leq \text{Vol}(M, g)^{\frac{1}{n}}.$$

The lowermost bound then follows by taking infimums in g . (Incidentally, this inequality will be strict unless the pure pieces are real hyperbolic.) For the uppermost bound, we note that the metrics g_j constructed above, when scaled to have curvatures bounds $-1 \leq K_{g_j} \leq 1$, have volumes tending to $2^n \text{Vol}(M_{\text{sym}}, g_{\text{sym}})$. Finally, we note that $\text{Vol}_K(M) \leq \text{Minvol}(M)$ in general from their definition. \square

Proof of Proposition 5. The fundamental groups of the pieces in item (1) of the definition of higher graph manifolds are relatively hyperbolic, it was shown by Dadarlat–Guentner [9] that such groups are uniformly embeddable into a Hilbert space. This in turn implies they have the Haagerup property. So the results of Higson–Kasparov [14] imply that these groups satisfy the Baum–Connes conjecture with coefficients.

The fundamental group Γ of any piece in item (3) of the definition of higher graph manifolds is an extension of a torsion free relatively-hyperbolic group, Γ' , with virtually nilpotent peripheral groups by a virtually nilpotent group Γ'' . Since Γ' satisfies the Baum–Connes conjecture and Γ'' has the Haagerup property, being amenable, it follows from Theorem 7.7 in [23] that Γ also satisfies the Baum–Connes conjecture.

The group $\pi_1(M)$ is a graph of groups (in the Bass–Serre sense) because in each piece the boundaries are π_1 -injective. The vertex groups are the fundamental groups of the pieces, and the edge groups are the virtually nilpotent fundamental groups of the boundaries. Therefore, by the main result of Oyono–Oyono [22] the group $\pi_1(M)$ satisfies the Baum–Connes conjecture with coefficients, which implies the original Baum–Connes conjecture. \square

We thank J.-F. Lafont for suggesting the strategy of the following proof and pointing us to the relevant references. The concept of *asymptotic dimension* of a group was introduced by M. Gromov in [13]. We refer the reader to [2] for definitions and examples.

Proof of Theorem 8. Let Z_i be a piece of the higher graph manifold M . To begin with we claim that $\pi_1(Z_i)$ has finite asymptotic dimension.

The infranilpotent factor N_i is finitely covered by a nilmanifold N'_i . It follows from results of Bell and Danishnikov, Corollaries 54 & 68 in [2], that the asymptotic dimension of $\pi_1(N_i)$ is finite.

The main result of Osin in [21] shows that in this situation the asymptotic dimension of $\pi_1(M_i)$ is finite. The total space Z_i of the fibration $N_i \hookrightarrow Z_i \rightarrow M_i$ fits into a short exact sequence

$$1 \rightarrow \pi_1(N_i) \rightarrow \pi_1(Z_i) \rightarrow \pi_1(M_i) \rightarrow 1.$$

A theorem that deals with the asymptotic dimension of these types of extensions of groups has been shown by Bell–Dranishnikov, see Theorem 63 of [2]. They show that the asymptotic dimension of $\pi_1(Z_i)$ is bounded from above by the sum of the asymptotic dimensions of $\pi_1(N_i)$ and $\pi_1(M_i)$. Therefore $\pi_1(Z_i)$ has finite asymptotic dimension.

Here we will again use that $\pi_1(M)$ is a graph of groups, with vertex groups $\pi_1(Z_i)$. Each vertex group $\pi_1(Z_i)$ has finite asymptotic dimension, so Theorem 77 in [2] implies that $\pi_1(M)$ has finite asymptotic dimension, as claimed.

An adaptation of the arguments of [10] to our higher graph manifold M show that M is aspherical. Each piece Z_i is a product of two aspherical manifolds, so it is aspherical. Induction on the number of pieces shows that M itself is aspherical, following the steps of Lemma 3.1 in [10].

It is now a consequence of the results of G. Yu in [28], that, since M is aspherical and $\pi_1(M)$ has finite asymptotic dimension, the manifold M does not admit any smooth metric of positive scalar curvature. \square

Proof of Corollary 9. If a smooth compact manifold M admits a volume collapsing sequence of metrics with bounded curvature then $\mathcal{Y}(M) \geq 0$ [24]. On the other hand, a well known fact about the Yamabe invariant is that $\mathcal{Y}(M) > 0$ if and only if M admits a smooth metric

of positive sectional curvature, (we refer again to [24]). Therefore Theorems 1 and 8 imply $\mathcal{Y}(M) = 0$. \square

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